A double integral of a product

1184. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*
Evaluate the following integral:

$$\int_0^\infty \int_0^\infty \frac{\sin x \sin(x+y)}{x(x+y)} dx dy.$$

Solution by the Missouri State University Problem Solving Group.

We will show that, more generally,

$$\int_0^\infty \int_0^\infty f(x) f(x+y)\, dy\, dx = \frac{1}{2}\left(\int_0^\infty f(t)\, dt\right)^2.$$

Since it is well known that

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2},$$

the value of the original integral is $\pi^2/8$.

Letting $u = x$ and $v = x + y$, reversing the order of integration, and then reversing the roles of u and v, we have

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty f(x) f(x+y)\, dy\, dx \\ &= \int_0^\infty \int_u^\infty f(u) f(v)\, dv\, du \\ &= \int_0^\infty \int_0^v f(u) f(v)\, du\, dv \\ &= \int_0^\infty \int_0^u f(u) f(v)\, dv\, du. \end{aligned}$$

Therefore

$$\begin{aligned} 2I &= \int_0^\infty \int_u^\infty f(u) f(v)\, dv\, du + \int_0^\infty \int_0^u f(u) f(v)\, dv\, du \\ &= \int_0^\infty \int_0^\infty f(u) f(v)\, dv\, du \end{aligned}$$

$$= \left(\int_0^\infty f(t)\, dt \right)^2 ,$$

and the result follows.

We note that similar techniques show that

$$\int_0^\infty \cdots \int_0^\infty f(x_1) f(x_1 + x_2) \ldots f(x_1 + x_2 + \ldots + x_n)\, dx_n \ldots\, dx_1$$

$$= \frac{1}{n!} \left(\int_0^\infty f(t)\, dt \right)^n .$$

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